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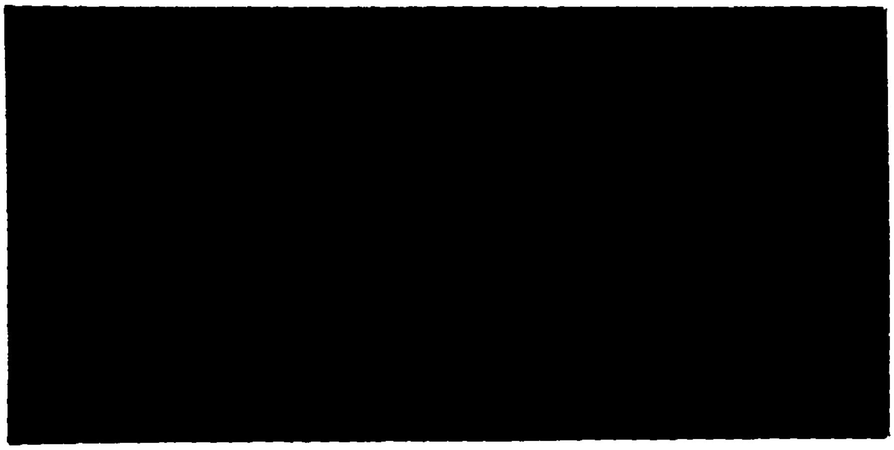

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Page 10, last line. "qQ, + tT"

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LA-1891

SOLUTION OF THE TRANSPORT EQUATION BY S_n APPROXIMATIONS

by

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This report is a revision of LA-1599.

PHYSICS

ABSTRACT

A numerical method for solving diffusion problems in plane, spherical, and cylindrical geometries is developed in this report. The method is, for the sake of brevity, referred to as the S_n-method. Specifically, it applies to the integro-differential equation of Boltzman, known in neutron diffusion work as the Transport Equation. Solutions are obtained in the spherical case for the stationary as well as the time-dependent form of the equation. Moreover, it is shown that problems in plane and cylindrical geometries may be identified with problems relating to spheres.

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1. Introduction.

Since the first report on the S_n-method[1] was issued, the method has been simplified and improved in a number of ways. This applies above all to the time-dependent case. The method has also been extended to handle problems of a more elaborate nature. These may now, for instance, involve anisotropic scattering or diffusion in cylindrical systems. For these and other reasons, a revised and amplified report is called for.

The S_n-method has, to date, been applied to a wide variety of problems, too numerous to discuss in detail. The report will therefore be confined to general or standard types of problems. Once the general features of the method are understood, adaptation to particular problems is not difficult. The method may, in any case, be regarded as thoroughly tested in practice. It has been found to be efficient, dependable, and accurate, and is therefore recommended for general use. Certain classes of problems permit a simplification of the Transport Equation. Even these may be more readily solved by the S_n-method than by the methods now in use.

1) LA-1599, October 1953.

2. The Transport Equation[2].

For the case of spherical geometry, isotropic scattering, and stationary problems, and with neutrons classified in velocity groups, the Transport Equation has the following form:

$$(1) \qquad \left[\mu D_r + \frac{1-\mu^2}{r} D_\mu + \sigma\right] N(r,\mu) = S(r),$$

where the source term $S(r)$ is given by (3), and the bracket denotes a partial differential operator. $N(r,\mu)$ represents the neutron flux (neuts/cm^2 sec) at the radial distance r (cm) in the direction θ ($\mu = \cos\theta$) with respect to the positive r-direction. The total cross section σ(1/cm), the source term $S(r)$, and the flux $N(r,\mu)$ depend in general on the velocity group. This dependence will be indicated by a subscript g whenever clarity demands it. In addition, σ usually depends on r. σ is, for instance, a step function of r if several materials are involved.

In the time-dependent case, equation (1) is replaced by:

$$(2) \qquad \left[\frac{1}{v} D_t + \mu D_r + \frac{1-\mu^2}{r} D_\mu + \sigma\right] N(t,r,\mu) = S(t,r),$$

where t denotes time (sec) and v_g neutron velocity (cm/sec).

Equations and formulae for the plane case are obtained by merely removing the terms in 1/r wherever they occur, and

[2] For derivations and general background, the following treatise is recommended: K. M. Case, F. de Hoffmann, and G. Placzek; Introduction to the Theory of Neutron Diffusion, Volume I, United States Government Printing Office, Washington, D. C., 1953.

interpreting r = 0 as some origin plane. A separate discussion of the plane case is therefore not necessary.

3. The Source Term S.

In the isotropic case the source or coupling term S(t,r) does not depend on μ. The standard form of S(t,r) is therefore quite simple. It is given by:

$$(3) \quad S_g(t,r) = \sum_{g'} \sigma_{gg'} N_{g'}(t,r) \equiv \frac{1}{2} \sum_{g'} \sigma_{gg'} \int_{-1}^{1} N_{g'}(t,r,\mu)d\mu,$$

where t is to be omitted in the stationary case. The transfer coefficients $\sigma_{gg'}$ (1/cm) represent the number of neutrons transferred (per cm) from group g' to group g, and may, like σ_g, depend on r. The relationship between $\sigma_{gg'}$ and actual cross sections, measured or estimated, is illustrated below.

Consider the following idealized neutron-nuclei interactions with the indicated cross sections given: (A) Absorption, $\sigma_{g'}^a$ given. (B) Elastic or velocity-preserving scattering, $\sigma_{g'}^e$ given. (C) Inelastic or velocity-degrading scattering[3] $(v_{g'} \rightarrow v_g)$, $\sigma_{g'}^i$ and the scattering spectrum $\eta_{gg'}$ given, with $\sum_g \eta_{gg'} = 1$, and $v_g \leq v_{g'}$. (D) Emission of ν neutrons due to fission, $\sigma_{g'}^f$ and the fission spectrum ν_g (independent of g')

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[3] One usually distinguishes between two types: a) true inelastic scattering with no coupling of v_g and the scattering angle, and b) slowing-down scattering with such coupling.

given, with $\sum_g \nu_g = \nu$. Of these interactions, elastic scattering and inelastic scattering of the slowing-down type may be anisotropic processes, that is, $\sigma_{g'}^{e}$ and $\sigma_{g'}^{i}$ may depend on the deflection cosine $\bar{\mu}$, $-1 \leq \bar{\mu} \leq 1$. In the case of slowing-down, a range of $\bar{\mu}$ is, furthermore, associated with each v_g.

For the case illustrated, the total cross section $\sigma_{g'}$ and the transfer coefficients $\sigma_{gg'}$ are, as the terminology suggests, given by:

(4) $$\sigma_{g'} = \sigma_{g'}^{a} + \sigma_{g'}^{e} + \sigma_{g'}^{i} + \sigma_{g'}^{f}, \text{ and}$$

(5) $$\sigma_{gg'} = \sigma_{g'}^{e} \delta_{gg'} + \sigma_{g'}^{i} \eta_{gg'} + \sigma_{g'}^{f} \nu_g,$$

where $\delta_{gg'} = 1$ if $g = g'$ and zero if $g \neq g'$.

4. <u>The Transport Approximation.</u>

If $\sigma_{g'}^{e}$ and $\sigma_{g'}^{i}$, introduced above, depend on $\bar{\mu}$ we have no longer an isotropic problem and equation (3) is not valid. It may, however, be regarded as approximately valid, provided certain parameter modifications are made. This procedure, which has a certain amount of theoretical justification, is known as the Transport Approximation.

The following modifications usually are made: (A) $\sigma_{g'}^{e}$ and $\sigma_{g'}^{i}$ are replaced by the corresponding integrals over $\bar{\mu}$. In particular, this applies to the expressions (4) and (5). (B) Certain parameter corrections, $\varepsilon_{g'}$, are then calculated and subtracted from $\sigma_{g'}$ and $\sigma_{g'g'}$, where $\epsilon_{g'}$ is obtained from:

$$(6) \qquad \epsilon_{g'} = \int_{-1}^{1} \bar{\mu} \left[\sigma_{g'}^{e} (\bar{\mu}) + \sigma_{g'}^{i} (\bar{\mu}) \right] d\bar{\mu}.$$

Although the Transport Approximation is essentially a recipe, it is commonly used and quite accurate, at least in the integral sense. In doubtful cases, the anisotropic problem should be solved directly, using the methods of Section 9.

5. Definition of the S_n-Method.

We divide the μ-interval (-1,1) into n intervals (μ_{j-1}, μ_j), $j=1,2,\ldots,n$, $\mu_0 = -1$, $\mu_n = 1$, and approximate $N(r,\mu)$ by n connected straight line segments as follows:

$$(7) \qquad N(r,\mu) = \frac{\mu - \mu_{j-1}}{\mu_j - \mu_{j-1}} N(r,\mu_j) + \frac{\mu_j - \mu}{\mu_j - \mu_{j-1}} N(r,\mu_{j-1}),$$

where $\mu_{j-1} \leq \mu \leq \mu_j$. The order of the approximation, denoted by n, may be any positive integer. Equation (7) is used below to transform equation (1), which involves the function $N(r,\mu)$, into n equations relating the functions $N(r,\mu_j)$, $j=0,1,\ldots,n$. The transformation is accomplished by substituting (7) in (1) and then integrating both sides of (1) over μ, separately for each μ-interval. An additional equation is obviously required and this is obtained by substituting $\mu = -1$ directly in (1). The reason for this choice will be apparent after reading Section 6.

The μ-mesh defined above will be referred to as standard if the length ℓ_j of (μ_{j-1}, μ_j) is equal to 2/n for all j. It may, however, be chosen to suit the particular problem one is considering. The choice of n may also be related to the application intended. Experience has shown, however, that the S_4-approximation satisfies most accuracy requirements. Large values of n are, in any case, not necessary; in fact, n = 2 or 3 may often do.

To facilitate the transformation referred to, we evaluate first a number of integrals involving (7). The range of integration is, in each case, the interval (μ_{j-1}, μ_j). To simplify the notation, we replace $N(r, \mu_j)$ by N(j) and $N(r, \mu_{j-1})$ by M(j). We have:

$$(8) \qquad \begin{cases} \int N(r,\mu)\, d\mu = \frac{1}{2}\, \ell_j \left[N(j) + M(j)\right] , \\ \int \mu N(r,\mu)\, d\mu = \frac{1}{2}\, \ell_j \left[a_j N(j) + \bar{a}_j M(j)\right] , \\ \int (1-\mu^2)\, D_\mu\, N(r,\mu)\, d\mu = \frac{1}{2} \ell_j b_j \left[N(j) - M(j)\right] , \end{cases}$$

where $a_j = (2\mu_j + \mu_{j-1})/3$, $\bar{a}_j = (\mu_j + 2\mu_{j-1})/3$, and

$b_j = (2/3)(3 - \mu_j^2 - \mu_j \mu_{j-1} - \mu_{j-1}^2)/(\mu_j - \mu_{j-1})$. We assume in what follows that $a_j \neq 0$ in order to simplify the discussion.

Now, performing the transformation, we obtain the following equations, the S_n-equations for the stationary case:

$$(9) \qquad (aD_r + \frac{b}{r} + \sigma)\, N + (\bar{a}D_r - \frac{b}{r} + \sigma)\, M = cS,$$

where a, $\bar{a}$, and b depend on j, j = 1,2,...,n, N and M on j and r, and S on r, and where $c \leqq 2$. The equation for $N(0)$, obtained by letting $\mu = -1$ in (1), may be included in (9) if for $j = 0$ we let $a = -1$, $b = M = 0$, and $c = 1$.

From (7) it also follows that the total flux $N(r)$ is given by:

$$(10) \qquad N(r) \equiv \frac{1}{2} \int_{-1}^{1} N(r,\mu)\, d\mu = \sum_{j=0}^{n} p_j\, N(j),$$

where $p_0 = \ell_1/4$, $p_n = \ell_n/4$, and $p_j = (\ell_j + \ell_{j+1})/4$, $j = 1,2,\ldots, n-1$, which in the standard case reduces to $p_0 = p_n = 1/2n$, and $p_j = 1/n$.

The above procedure may be duplicated for the time-dependent case, with the result that (9) is replaced by:

$$(11) \qquad \left(\frac{1}{v} D_t + aD_r + \frac{b}{r} + \sigma\right) N + \left(\frac{1}{v} D_t + \bar{a}D_r - \frac{b}{r} + \sigma\right) M = cS,$$

where N, M, and S now depend on t as well.

In line with remarks made in Section 3, the plane case now may be identified with the spherical case by simply letting $b = 0$ in (9) and (11).

6. Integration of the S_n-Equations.

To obtain accuracy in the numerical integration of equations describing particle flow, it is usually necessary to integrate in the direction of the flow. Consider a thin layer of material and, incident on it, a stream of particles of intensity Q_1. The emerging intensity Q_2 may then be equated to $qQ_1 +$ [illegible]T, where q is the attenuation factor ($q < 1$) and T the source term (the intensity contributed by the layer of material). The errors in q and T are essentially controlled by making the integration steps small enough, the error in Q_1 by the fact that $q < 1$. On the other hand, if Q_1 is obtained from Q_2, i.e., $Q_1 = (Q_2 - T)/q$, we can expect, in general, the division to amplify the errors.

The integration rules to be used in connection with (9) and (11) are based on the above observation.

Consider first the integration over μ. It is readily verified that streaming in a spherical system is associated with an increase in μ. A stream directed towards the center of the system is the only exception. In this case, μ jumps from -1 to 1 at the origin. The integration over μ will therefore be performed in the positive μ-direction starting at $\mu = -1$. This is the reason for incorporating equations involving $N(r,-1)$ in (9) and (11).

Consider next the integration over r. If a_j is negative, we are concerned with particles directed toward the central parts of the system. If, on the other hand, a_j is positive, then outward-directed particles are involved. The integration over r will therefore be performed in the negative direction if $a_j < 0$ and in the positive direction if $a_j > 0$.

The complete integration procedure can now be outlined. We assume that at some stage of the calculation $N(r)$ and hence $S(r)$

are given for each g. Equation (9) is then integrated over r, according to the above rules, for $j=0,1,\ldots,n$ and in that order. This is repeated for each g and finally (10) is used to obtain a new set of N(r)-functions. In the stationary case this procedure establishes an iteration method for arriving at the solution of (1). In the time-dependent case the procedure is used to advance the solution from one time step to the next, that is, to solve equation (2).

7. The Stationary Case.

We divide the r-interval $(0,r_I)$, where r_I is the outer radius of the system, into I intervals (r_{i-1},r_i), $i=1,2,\ldots,I$, $r_0 = 0$, and approximate any function F defined over (r_{i-1},r_i) by a straight line. The integrals of D_rF and F over the interval are then given by $F_i - F_{i-1}$ and $\frac{1}{2}\Delta_i(F_i + F_{i-1})$, respectively, where Δ_i is the length of the ith interval. The parameters σ_g and $\sigma_{gg'}$ will be specified for each interval rather than its bounds, and r in the D_μ-term will be treated in the same way, i.e., r will be replaced by $s_i = \frac{1}{2}(r_i + r_{i-1})$. Integrating equation (9) over r as indicated above, letting $i-1 = k$, $m = |a|$, and $\bar{m} = \bar{a}$ sign (a), we obtain for $a_j > 0$:

$$(12)\quad m(N_i-N_k) + \left(\frac{\sigma_i\Delta_i}{2} + \frac{b\Delta_i}{2s_i}\right)(N_i+N_k) + \bar{m}(M_i-M_k)+\left(\frac{\sigma_i\Delta_i}{2} - \frac{b\Delta_i}{2s_i}\right)(M_i+M_k)= \frac{c\Delta_i}{2}(S_i+S_k),$$

or solving for N_i:

$$(13)\quad N_i = \frac{(m - \frac{\sigma_i\Delta_i}{2} - \frac{b\Delta_i}{2s_i})N_k + (\bar{m} - \frac{\sigma_i\Delta_i}{2} + \frac{b\Delta_i}{2s_i})M_k - (\bar{m} + \frac{\sigma_i\Delta_i}{2} - \frac{b\Delta_i}{2s_i})M_i + \frac{c\Delta_i}{2}(S_i+S_k)}{(m + \frac{\sigma\Delta_i}{2} + \frac{b\Delta_i}{2s_i})}.$$

Repeating the above steps for $a_j < 0$ with reference to the interval (r_i, r_{i+1}) and with $i+1 = k$ we have:

$$(14)\quad -m(N_k - N_i) + (\frac{\sigma_i\Delta_i}{2} + \frac{b\Delta_i}{2s_i})(N_k + N_i) - \bar{m}(M_k - M_i) + (\frac{\sigma_i\Delta_i}{2} - \frac{b\Delta_i}{2s_i})(M_k + M_i) = \frac{c\Delta_i}{2}(S_k + S_i),$$

or solving for N_i:

$$(15)\quad N_i = \frac{(m - \frac{\sigma_i\Delta_i}{2} - \frac{b\Delta_i}{2s_i})N_k + (\bar{m} - \frac{\sigma_i\Delta_i}{2} + \frac{b\Delta_i}{2s_i})M_k - (\bar{m} + \frac{\sigma_i\Delta_i}{2} - \frac{b\Delta_i}{2s_i})M_i + \frac{c\Delta_i}{2}(S_k+S_i)}{m + \frac{\sigma\Delta_i}{2} + \frac{b\Delta_i}{2s_i}}$$

Note that (15) is identical to (13) in form except that σ_i, Δ_i, and s_i in the first case are associated with the interval (r_{i-1}, r_i) and in the second with (r_i, r_{i+1}).

Equation (13) or (15) represents the S_n-solution of equation (1). To initiate the calculations, the conditions at $r = r_I$ and $r = 0$ must be given. If $S(r)$ is given by (3) we let $N(r_I, \mu_j) = 0$ for $a_j < 0$, i.e., no inward flux at r_I, and let $N(0, \mu_j) = N(0, -\mu_j)$ for $a_j > 0$, i.e., continuity at the center. Furthermore, the $N(r)$-functions must be specified initially so that the $S(r)$-functions may be obtained. If better information is unavailable, they may be set identically equal to unity.

Finally some sort of convergence criterion must be applied to terminate the calculation. One may, for instance, compare two successive iterations in a point-wise manner (for each i) until the desired agreement is obtained.

The r-mesh is chosen, like the μ-mesh, to obtain the desired degree of accuracy. It need not be uniform and is essentially determined by the smoothness of the flux-functions. Experience has shown that from 20 to 50 space points are normally sufficient.

The tables below summarize the information necessary for S_2 and S_4 calculations if the μ-mesh is standard.

Table for S_2.

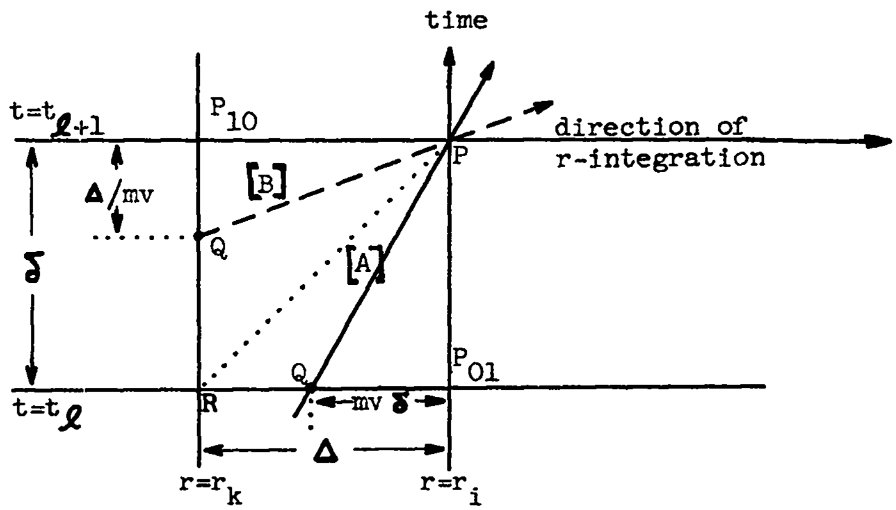
j	N	M	k	m	$\bar{m}$	b	c	p
0	N(0)	0	i+1	1	-	0	1	1/4
1	N(1)	N(0)	i+1	1/3	2/3	4/3	2	1/2
2	N(2)	N(1)	i-1	2/3	1/3	4/3	2	1/4

Table for S_4.

j	N	M	k	m	$\bar{m}$	b	c	p
0	N(0)	0	i+1	1	-	0	1	1/8
1	N(1)	N(0)	i+1	2/3	5/6	5/3	2	1/4
2	N(2)	N(1)	i+1	1/6	1/3	11/3	2	1/4
3	N(3)	N(2)	i-1	1/3	1/6	11/3	2	1/4
4	N(4)	N(3)	i-1	5/6	2/3	5/3	2	1/8

8. The Time-Dependent Case.

The solution of equation (11), the S_n-transform of equation (2), will require integration over the variables t and r. A set of time intervals $(t_{\ell}, t_{\ell+1})$ of length δ_{ℓ}, $\ell = 0,1,\ldots,$ must therefore be coupled with the r-mesh defined in Section 7. The integration method, to be described below, may be classed as a characteristic method. The characteristic direction is here identified with the direction of the particle flow in time and space. This is clearly determined by the coefficients of D_t and D_r in (11). Since the latter are constants, the slope of the characteristic QP (see diagram below) is, in fact, given by $1/m_j v$ where $m_j = |a_j|$. A typical integration step consists of finding the flux-functions N(j) at P in terms of information available at R, P_{01}, and P_{10} (obtained by previous integration steps).



Two cases arise and these will be considered separately. (A) The characteristic line QP intersects the line $t = t_\ell$. In this case we make the assumption that a function F defined over the triangle [A], RPP_{01}, may be approximated over [A] by a plane surface. (B) The line QP intersects the line $r = r_k$ ($k = i-1$ if $a_j > 0$, $k = i+1$ if $a_j < 0$). In this case we make the corresponding assumption with respect to the triangle [B], RPP_{10}.

Case A.

It will be convenient below to have equation (11) available in the following form:

$$(16) \quad (\tfrac{1}{v}D_t + aD_r)(N + M) = U \equiv -(\sigma + \tfrac{b}{r})N - (\sigma - \tfrac{b}{r})M + \bar{\beta}\, D_r M + cS,$$

where $\bar{\beta} = a - \bar{a}$. The assumption made permits us to evaluate D_t along $r = r_i$ ($P_{01}P$) and D_r along $t = t_\ell$ (QP_{01}) and this is equivalent to finding the total derivative $\frac{1}{v}D_t + aD_r$ along QP. The latter may then be thought of as centered at the midpoint of QP.

For the sake of consistency, U should also be centered at the midpoint of QP, that is, replaced by $\frac{1}{2}(U_Q + U_P)$. U is, however, constant over the triangle [A] since the derivative of the left-hand side of (16) vanishes by assumption. How U is to be averaged is therefore determined by other considerations. Primary among these is the desire to have the resulting difference equation conform with the stationary case if $D_t = 0$, which leads to the conclusion that U should be centered on the diagonal RP and hence be replaced by $\frac{1}{2}(U_R + U_P)$.

The directions given above are now applied to equation (16) to transform (16) into a difference equation. In what follows, t_ℓ-dependence will be indicated by a superscript 0 and $t_{\ell+1}$-dependence by a superscript 1. The difference version of (16) is given by:

$$(17)\quad \frac{1}{v\delta_0}(N_i^1+M_i^1-N_i^0-M_i^0)+ \frac{m}{\Delta_i}(N_i^0+M_i^0-N_k^0-M_k^0) = -(\frac{\sigma_i}{2} + \frac{b}{2s_i})(N_i^1+N_k^0)$$

$$-(\frac{\sigma_i}{2} - \frac{b}{2s_i})(M_i^1+M_k^0) + \frac{\beta}{2\Delta_i}(M_i^1+M_i^0-M_k^1-M_k^0)+ c(S_i^1+S_k^1),$$

where $\beta = m - \bar{m}$. Adding $\frac{m}{\Delta_i}(N_k^0+M_k^0-N_i^1-M_i^1)$ on both sides of (17) and multiplying all terms by $\Delta_i w$, where $w = mv\delta_0/\Delta_i$, we have:

$$(18)\quad m(1-w)(N_i^1+M_i^1-N_i^0-M_i^0) = -w(m + \frac{\sigma_i\Delta_i}{2} + \frac{b\Delta_i}{2s_i})\, N_i^1 + wK,$$

where K is given by:

$$(19)\quad K = (m- \frac{\sigma_i\Delta_i}{2} - \frac{b\Delta_i}{2s_i})N_k^0+(m- \frac{\sigma_i\Delta_i}{2} + \frac{b\Delta_i}{2s_i})M_k^0-(m+ \frac{\sigma_i\Delta_i}{2} - \frac{b\Delta_i}{2s_i})M_i^1$$

$$+ \frac{\beta}{2}(M_i^1+M_i^0-M_k^1-M_k^0) + \frac{c\Delta_i}{2}(S_i^1 + S_k^0).$$

Finally, solving (18) for N_i^1 we obtain:

$$(20) \quad N_i^1 = \frac{m(1-w)\left[N_i^0 + M_i^0 - M_i^1\right] + wK}{m(1-w) + w(m + \frac{\sigma_i \Delta_i}{2} + \frac{b\Delta_i}{2s_i})} \quad .$$

<u>Case B.</u>

The assumption made in this case permits us to evaluate D_r along $t = t_{\ell+1}$ $(P_{10}P)$ and D_t along $r = r_k$ (QP_{10}), that is, the total derivative $\frac{1}{v}D_t + aD_r$ along QP. The function U is handled as in Case A, and we obtain, corresponding to (17):

$$(21) \quad \frac{1}{v\delta_0}(N_k^1+M_k^1-N_k^0-M_k^0) + \frac{m}{\Delta_i}(N_i^1+M_i^1-N_k^1-M_k^1) = \frac{1}{2}(U_i^1+U_k^0),$$

where $\frac{1}{2}(U_i^1+U_k^0)$ is identical to the right-hand side of (17). Adding $\frac{m}{\Delta_i}(N_k^0+M_k^0-N_i^1-M_i^1)$ to both sides of (21), and multiplying both sides by Δ_i letting $w = \Delta_i/mv\delta_0$, we have:

$$(22) \quad m(1-w)(N_k^0+M_k^0-N_k^1-M_k^1) = -(m + \frac{\sigma_i\Delta_i}{2} + \frac{b\Delta_i}{2s_i})\ N_i^1 + K.$$

Finally, solving (22) for N_i^1 we obtain:

$$(23) \quad N_i^1 = \frac{m(1-w)(N_k^1+M_k^1-N_k^0-M_k^0) + K}{m + \frac{\sigma_{i\ i}}{2} + \frac{b\Delta_i}{2s_i}} \quad .$$

Equations (20) and (23) represent the S_n-solution of (2) in the time-dependent case. To perform the calculations the flux functions must be given at $t = t_0$, the starting time. The conditions at $r = 0$ and $r = r_I$ are, in the standard case, identical to those given for the stationary equation. Note that in evaluating K above, S_i^1 is not available. It is therefore, as a rule, necessary to iterate once for each time step, taking $S_i^1 = S_i^0$ for the first integration, and then repeating, with the values of S_i^1 thus obtained, the calculations for this time step.

9. The Anisotropic Case.

In the case of stationary problems and anisotropic scattering, the source term S(r) as given by (3) is replaced by:

$$(24) \quad S(r,\mu) = \sum_{g'} \sigma_{gg'} N(r) + \sum_{g'} T_{gg'}(r,\mu)$$

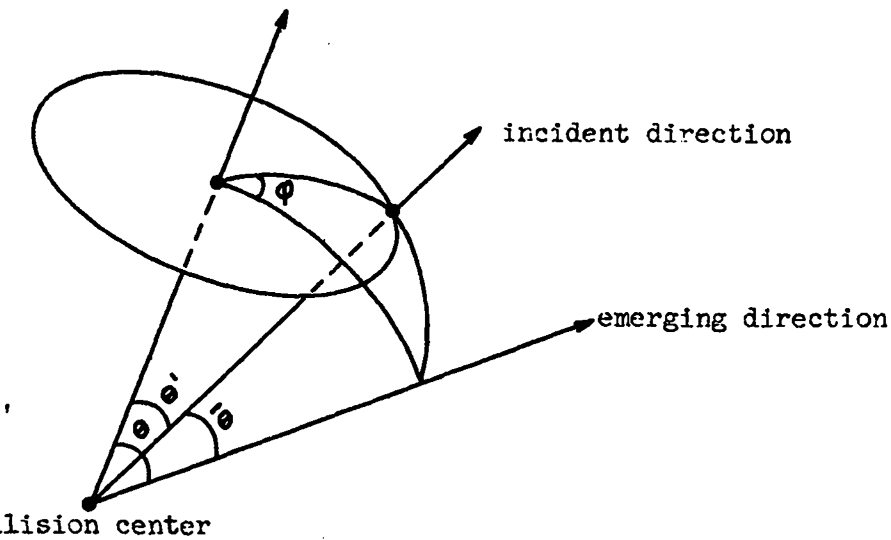
where

$$(25) \quad T_{gg'}(r,\mu) = \frac{1}{\pi} \int_{-1}^{1} \int_{0}^{\pi} N_{g'}(r,\mu') \left[\sigma_{gg'}(\bar{\mu}) - \sigma_{gg'} \right] d\phi \, d\mu',$$

which expresses the relation between the flux $N_{g'}(r,\mu')$ prior to interactions and the resulting emergent flux $T_{gg'}(r,\mu)$. The quantity $\sigma_{gg'}$ represents the integral of $\sigma_{gg'}(\bar{\mu})$ over $\bar{\mu}$, $-1 \leq \bar{\mu} \leq 1$. From the diagram below we observe that the following relation between $\bar{\mu}$, μ', and ϕ holds:

$$(26) \quad \bar{\mu} = \mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)} \cos \phi .$$

$\mu = \cos\theta$
$\mu' = \cos\theta'$
$\bar{\mu} = \cos\bar{\theta}$



To evaluate $T_{gg'}(r,\mu)$ we expand $\left[\sigma(\bar{\mu}) - \sigma\right]_{gg'}$ and $N_{g'}(r,\mu')$ in Legendre series:

(27) $$\left[\sigma(\bar{\mu}) - \sigma\right]_{gg'} = \frac{1}{2}\left[3a^1_{gg'}P_1(\bar{\mu}) + 5a^2_{gg'}P_2(\bar{\mu}) + \text{-----}\right], \text{ and}$$

(28) $$N_{g'}(r,\mu') = \frac{1}{2}\left[N_{g'}(r) + 3N^1_{g'}(r)P_1(\mu') + 5N^2_{g'}(r)P_2(\mu') + \text{---}\right],$$

where $a^m_{gg'}$ and $N^m_{g'}(r)$ are given by:

(29) $$a^m_{gg'} = \int_{-1}^{1}\left[\sigma(\bar{\mu}) - \sigma\right]_{gg'}P_m(\bar{\mu})d\bar{\mu}, \text{ and}$$

(30) $$N^m_{g'}(r) = \int_{-1}^{1} N_{g'}(r,\mu')\, P_m(\mu')d\mu' .$$

Since $N(r,\mu')$ is approximated by a set of connected straight line segments in μ, it is clear that $N^m_{g'}(r)$ may be expressed in terms of the S_n-functions $N_{g'}(j) \equiv N_{g'}(r,\mu_j)$.

With the aid of the above expansions and the addition theorem associated with Legendre polynomials, the series for $T_{gg'}(r,\mu)$ can be obtained. This series is given by:

$$(31) \quad T_{gg'}(r,\mu) = \frac{1}{2}\left[3a^1_{gg'}N^1_{g'}P_1(\mu) + 5a^2_{gg'}N^2_{g'}P_2(\mu) + ----\right].$$

If now the integrations over (μ_{j-1},μ_j) associated with the S_n-method are performed upon (31), we obtain a set of $T_{gg'}(j)$. The source term may therefore be written as follows:

$$(32) \quad S(j) = \sum_{g'} \left[\sigma_{gg'}N_{g'} + T_{gg'}(j)\right] ,$$

where $T_{gg'}(j)$ represents the anisotropic contribution.

To obtain the explicit expression for $T_{gg'}(j)$, it is necessary to perform a number of simple integrations over μ, first to find $N^m_{g'}$ in terms of $N_{g'}(j)$ and then the indicated sub-integrals of (31). In the standard case with n = 2 and 4 we have in matrix notation:

S_2-approximation:

$$\begin{pmatrix} T_{gg'}(1) \\ T_{gg'}(2) \\ T_{gg'}(3) \end{pmatrix} = \begin{pmatrix} -1.0 & 1 \\ -.5 & 0 \\ .5 & 0 \end{pmatrix} \begin{pmatrix} a^1_{gg'} & 0 \\ 0 & a^2_{gg'} \end{pmatrix} \begin{pmatrix} -.5000 & .0000 & .5000 \\ .3125 & -.6250 & .3125 \end{pmatrix} \begin{pmatrix} N_{g'}(0) \\ N_{g'}(1) \\ N_{g'}(2) \end{pmatrix}$$

S_4-approximation:

$$\begin{pmatrix} T_{gg'}(1) \\ T_{gg'}(2) \\ T_{gg'}(3) \\ T_{gg'}(4) \\ T_{gg'}(5) \end{pmatrix} = \begin{pmatrix} -1.00 & 1.000 & -1.0000 & 1.0000 \\ -.75 & .375 & -.0469 & -.1172 \\ -.25 & -.375 & .2969 & .1172 \\ .25 & -.375 & -.2969 & .1172 \\ .75 & .375 & .0469 & -.1172 \end{pmatrix} \begin{pmatrix} a^1_{gg'} & 0 & 0 & 0 \\ 0 & a^2_{gg'} & 0 & 0 \\ 0 & 0 & a^3_{gg'} & 0 \\ 0 & 0 & 0 & a^4_{gg'} \end{pmatrix}$$

$$\times \begin{pmatrix} -.3125 & -.3750 & .0000 & .3750 & .3125 \\ .3516 & -.0781 & -.5470 & -.0781 & .3516 \\ -.2461 & .4922 & .0000 & -.4922 & .2461 \\ .0791 & -.3457 & .5332 & -.3457 & .0791 \end{pmatrix} \begin{pmatrix} N_{g'}(0) \\ N_{g'}(1) \\ N_{g'}(2) \\ N_{g'}(3) \\ N_{g'}(4) \end{pmatrix}$$

10. Cylindrical Geometry[4].

The S_n-method can also be applied to cylindrical systems, finite or infinite, which are symmetric about a given axis. Furthermore, with proper choice of coordinates, problems in cylindrical geometry can, in most cases, be identified with problems relating to spheres. For the sake of simplicity, we limit the discussion below to systems also symmetric about an origin plane, perpendicular to the axis.

The following choice of coordinates is made: Position in the system is denoted by (z,r) where z is the distance from the origin plane and r the distance from the axis. The direction of a neutron beam is specified by a deflection angle φ ($\eta = \cos\varphi$) and an azimuthal angle θ ($\mu = \cos\theta$). The two angles φ and θ are measured with respect to the axial (z) and radial (r) directions, the latter angle in a plane parallel to the origin plane.

In terms of the above coordinates, the Transport Equation for the isotropic case is given by:

$$(33) \qquad \left[\frac{1}{v}D_t + \eta D_z + \sqrt{1-\eta^2}\left(\mu D_r + \frac{1-\mu^2}{r} D_\mu\right) + \sigma\right] N = S$$

where $\frac{1}{4\pi} N d\Omega \equiv \frac{1}{4\pi} N(t,z,r,\eta,\mu) d\Omega$ is the total neutron flux

[4]In collaboration with Stewart Schlesinger.

(neuts/cm^2 sec) over the angular differential element $d\Omega \equiv \sin\varphi\, d\varphi\, d\theta$, and $S \equiv S(t,r,z)$ is the source term with

$$(34)\quad S = \sum_{g'} \sigma_{gg'} N_{g'} = \frac{1}{2\pi} \sum_{g'} \sigma_{gg'} \int_{-1}^{1}\int_{-1}^{1} \frac{N_{g'}}{\sqrt{1-\mu^2}}\, d\eta\, d\mu \,.$$

The acceptability of η and μ as variables describing the direction of the beam (i.e., that both have ranges which are in one-to-one correspondence with the effective range of the angles of which they are the cosines) follows from the fact that the two angles describe the physical situation completely with ranges of only one-half of a revolution. This is clear for η by its very definition, and follows for μ due to symmetry about any radius in a plane perpendicular to the axis of the cylinder.

If one considers the time-independent problem for a finite cylinder, the transport equation becomes

$$(35)\qquad \left[\eta D_z + \sqrt{1-\eta^2}\,(\mu D_r + \frac{1-\mu^2}{r} D_\mu) + \sigma\right] N = S.$$

For values of η which are not equal to either one or minus one, it is clear that (35) can be written like a multi-velocity time-dependent problem for a sphere,

$$(36)\qquad \left[\frac{\eta}{\sqrt{1-\eta^2}} D_z + (\mu D_r + \frac{1-\mu^2}{r} D_\mu) + \frac{\sigma}{\sqrt{1-\eta^2}}\right] N = \frac{S}{\sqrt{1-\eta^2}} \,,$$

the solution of which has already been described in previous sections.[5)] For $\eta = \pm 1$, the problem is equivalent to that of a plane, for which solution by the S_n-method has previously been discussed.

The analogy described above between (36) and the time-dependent spherical problem might at first glance appear to break down when η becomes negative. However, in these cases the boundary conditions are known on the opposite circular face as for positive η. Hence, the variable z can be replaced by -z, and the boundary condition can then be taken as corresponding to the same circular face for all η. Therefore, one can advance through the (r,z) mesh in the same direction for each η (since the cylinder is assumed to be symmetric about an origin plane), in precisely the same manner as for a multi-velocity time-dependent problem for a sphere.

It should be noted that employing the above method of integration for (36), the sign of η does not influence numerical calculation provided the source term is assumed to be known for

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5) Due to the presence of the factor $1/\sqrt{1-\mu^2}$ in the integrand of the source term (34), the p_j-values (see p. 7) must be re-evaluated. For a standard μ-mesh, we obtain in the cylindrical case $p_0 = .3183$, $p_1 = .3634$, $p_2 = .3183$ for S_2, and $p_0 = .2180$, $p_1 = .2006$, $p_2 = .1628$, $p_3 = .2006$, $p_4 = .2180$ for S_4, rather than the values given in the tables on p. 14.

each stage of the integration. However, the sign of η must be taken into account when the source term is evaluated prior to the next stage of integration.

To evaluate the source term as indicated in (34), it is convenient to select the roots of a Legendre polynomial as the finite set of η values so that the Gauss Quadrature Formula can be employed. Since these roots are symmetric about zero, the advancing through the (r,z) mesh can be accomplished by using only positive η and then modifying the source term computation appropriately.

Another cylindrical problem which admits the same type of solution is that of the time-dependent infinite cylinder. The Transport Equation in such a case has the following form:

$$(37) \qquad \left[\frac{1}{v}D_t + \sqrt{1 - \eta^2}\,\left(\mu D_r + \frac{1 - \mu^2}{r} D_\mu\right) + \sigma\right] N = S.$$

Such an equation is directly equivalent to a time-dependent multi-velocity spherical problem, and the relationship to such a problem is obtained by merely increasing the number of velocity groups m-fold if one employs m different values of η.

It is clear that the sign of η does not influence (37). Unlike the case of the finite cylinder, the source term for the infinite cylinder can be computed without regard to the sign of η since the flux is symmetric about any plane perpendicular to the axis of the infinite cylinder. Hence the solution of (37) can be obtained dealing exclusively with positive values of η.

For the time-independent case of the infinite cylinder, the problem again is equivalent to a multi-velocity spherical problem, in an exactly analagous manner as for the time-dependent case.

In order to achieve accuracy consistent with that of the S_n-method, one should choose for the values of η the roots of the Legendre polynomial of order n+2, where n is the order of approximation in the S_n-method. This would be true for calculations relating to either a finite or infinite cylinder.

11. Comments on Computation.

The solutions obtained in this report are most effectively evaluated numerically with the aid of a modern electronic calculator. The time required is first, of course, a function of the speed of the computing equipment used. The calculations referred to in this report[6)] were performed on an IBM Type 701 calculator, characterized by a 60 μsec add time and a 450 μsec multiply time. The calculation time is also a function of the order of approximation (n), the number of radial points (I), and the number of groups

[6)] The flow diagrams and programs for the stationary as well as the time-dependent case, in their most general form, were prepared by Janet Bendt.

(G), in fact, approximately linear in these quantities. A typical stationary S_4-calculation involving one velocity group and 25 radial points, performed on the calculator mentioned, requires about one second per iteration. If the initial total flux is taken to be uniform over the entire system, or if the circumstances are otherwise unfavorable, about 25 iterations may be necessary. In the time-dependent case the same calculation takes about two seconds per time step.